Multiplicative autocorrelation in stationary Markov processes

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Abstract

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QUESTION: For any stationary markov process, is the autocorrelation of an interval the product of the autocorrelations of subintervals?

Summary

A stationary process Z_t has *multiplicative autocorrelation* when

$$\operatorname{Cor}[Z_t, Z_r] = \operatorname{Cor}[Z_t, Z_s] \operatorname{Cor}[Z_s, Z_r]$$

for all $t \leq s \leq r$. Autocorrelation is defined as

$$\operatorname{Cor}[Z_t, Z_s] := \frac{\operatorname{Cov}[Z_t, Z_s]}{\sigma^2}$$

with $\sigma^2 = \operatorname{Var}(Z_t)$.

A stationary autoregressive process has multiplicative autocorrelation [1]. However, not all stationary Markov processes have multiplicative autocorrelation. See the section below about a real-valued 3-state Markov chain for a counterexample.

Among discrete-time stationary processes, only autoregressive processes have multiplicative autocorrelation. Some Markov processes are not obviously autoregressive processes even though technically they are. For example, all stationary real-valued two-state Markov chains are autoregressive (and thus also have multiplicative autocorrelation).

Multiplicative autocorrelation implies autoregression

Consider any real-valued discrete-time stationary Markov process Z'_t and translate it to $Z_t := Z'_t - \operatorname{E}[Z'_t]$ without loss of generality.

Let

$$\sigma^2 := \mathrm{Var}(Z_t)$$
$$\rho := \mathrm{Cov}[Z_t, Z_{t+1}]\,/\sigma^2$$

Multiplicative autocorrelation implies

$$\mathrm{Cor}[Z_t, Z_{t+n}] = \rho^n$$
$$\mathrm{Cov}[Z_t, Z_{t+n}] = \rho^n \sigma^2$$

Define what will be shown to be "white noise" of Z_t as autoregressive process:

$$\epsilon_t := Z_t - \rho Z_{t-1}$$

By convenient translation,

$$\begin{aligned} \mathrm{E}[Z_t] &= 0 \\ \epsilon_t &= 0 \\ \mathrm{Cov}[Z_t, Z_s] &= \mathrm{E}[Z_t Z_s] \\ \mathrm{E}\left[Z_t^2\right] &= \sigma^2 \\ \mathrm{E}[Z_t Z_{t+1}] &= \rho \end{aligned}$$

Consider any $n > 0$.

$$\begin{aligned} \mathrm{E}[\epsilon_t \epsilon_{t+n}] &= \mathrm{E}[(Z_t - \rho Z_{t-1})(Z_{t+n} - \rho Z_{t+n-1})] \\ &= \mathrm{E}[Z_t Z_{t+n}] + \rho^2 \mathrm{E}[Z_{t-1} Z_{t+n-1}] - \rho(\mathrm{E}[Z_t Z_{t+n-1}] + \mathrm{E}[Z_{t-1} Z_{t+n}]) \\ &= (1 + \rho^2)\rho^n \sigma^2 - \rho(\rho^{n-1}\sigma^2 + \rho^{n+1}\sigma^2) \\ &= 0 \end{aligned}$$

thus ϵ_t satisfies the "white noise" condition for expressing Z_t as the autoregressive process

$$Z_{t+1} = \rho Z_t + \epsilon_t$$

QED

Real-valued 2-state Markov chain

For any stationary two-state Markov chain [1] Z_t,

$$\mathrm{Cor}[Z_t, Z_0] = \mathrm{Cor}[Z_t, Z_s]\,\mathrm{Cor}[Z_s, Z_0]$$

Proof Let

$$q_1 := \mathrm{P}(Z_t = a_1)$$
$$q_0 := \mathrm{P}(Z_t = a_0)$$

Map Z_t to a more convenient

$$Y_t := \frac{Z_t - a_0}{a_1 - a_0}$$

Since Y_t only equals 0 or 1:

$$\mathrm{E}[Y_t] = \mathrm{E}\big[Y_t^2\big] = q_1$$

and thus

$$\mathrm{Var}(Y_t) = q_1 - q_1^2 = q_1 q_0$$

For convenience let

$$\begin{aligned} p_0 &:= \mathrm{P}(Y_1 = 0 \mid Y_0 = 1) \\ p_1 &:= \mathrm{P}(Y_1 = 1 \mid Y_0 = 0) \\ s &:= p_0 + p_1 \end{aligned}$$

Since Y_t is stationary, it follows that $q_i = p_i/s$ for $i \in \{0, 1\}$. In preparation for induction, assume

$$\begin{aligned} \mathrm{P}(Y_t = 1 \mid Y_0 = 1) &= q_1 + q_0(1-s)^t \\ \mathrm{P}(Y_t = 1 \mid Y_0 = 0) &= q_1 - q_1(1-s)^t \end{aligned}$$

It must follow that

$$\begin{aligned} \mathrm{P}(Y_{t+1} = 1 \mid Y_0 = 1) &= \mathrm{P}(Y_{t+1} = 1 \mid Y_1 = 1)(1 - p_0) + \mathrm{P}(Y_{t+1} = 1 \mid Y_1 = 0)p_0 \\ &= [q_1 + q_0(1-s)^t](1 - p_0) + [q_1 - q_1(1-s)^t]p_0 \\ &= q_1 + [q_0(1 - p_0) - q_1 p_0](1-s)^t \\ &= q_1 + [q_0(1 - p_0) - (1 - q_0)p_0](1-s)^t \\ &= q_1 + [q_0 - p_0](1-s)^t \\ &= q_1 + [q_0 - q_0 s](1-s)^t \\ &= q_1 + q_0(1-s)^{t+1} \end{aligned}$$

and

$$\begin{aligned} \mathrm{P}(Y_{t+1} = 1 \mid Y_0 = 0) &= \mathrm{P}(Y_{t+1} = 1 \mid Y_1 = 1)p_1 + \mathrm{P}(Y_{t+1} = 1 \mid Y_1 = 0)(1 - p_1) \\ &= [q_1 + q_0(1-s)^t]p_1 + [q_1 - q_1(1-s)^t](1 - p_1) \\ &= q_1 + [q_0 p_1 - q_1(1 - p_1)](1-s)^t \\ &= q_1 + [(1 - q_1)p_1 - q_1(1 - p_1)](1-s)^t \\ &= q_1 + [p_1 - q_1](1-s)^t \\ &= q_1 + [q_1 s - q_1](1-s)^t \\ &= q_1 - q_1(1-s)^{t+1} \end{aligned}$$

which completes induction, noting the base case of $t = 0$ is true.

Due to the convenient mapping to Y_t,

$$\begin{aligned}
\mathrm{E}[Y_t Y_0] &= \mathrm{P}(Y_t = 1 \mid Y_0 = 1)\,\mathrm{P}(Y_0 = 1) \\
&= (q_1 + q_0(1-s)^t)q_1 \\
&= q_1^2 + q_0 q_1 (1-s)^t
\end{aligned}$$

thus

$$\begin{aligned}
\mathrm{Cov}[Y_t, Y_0] &= \mathrm{E}[Y_t Y_0] - \mathrm{E}[Y_t]\mathrm{E}[Y_0] \\
&= q_1^2 + q_0 q_1 (1-s)^t - q_1^2 \\
&= q_0 q_1 (1-s)^t \\
\mathrm{Cor}[Y_t, Y_0] &= (1-s)^t
\end{aligned}$$

QED

Counterexample of Real-Valued 3-State Markov Chain

Let Z_t be a stationary Markov process such that

$$\mathrm{P}(Z_t = -1) = \mathrm{P}(Z_t = 0) = \mathrm{P}(Z_t = 1) = 1/3$$

$$\begin{aligned}
\mathrm{P}(Z_{t+1} = 0 \mid Z_t = -1) = 1/2 \\
\mathrm{P}(Z_{t+1} = 1 \mid Z_t = 0) = 1/2 \\
\mathrm{P}(Z_{t+1} = -1 \mid Z_t = 1) = 1/2
\end{aligned}$$

and for all $i \in \{-1, 0, 1\}$,

$$\mathrm{P}(Z_{t+1} = i \mid Z_t = i) = 1/2$$

Conveniently $\mathrm{E}[Z_t] = 0$, thus $\mathrm{Cov}[Z_t, Z_s] = \mathrm{E}[Z_t Z_s]$. For one time step we have

$$\begin{aligned}
\mathrm{P}(Z_1 = 1 \wedge Z_0 = 1) &= (1/3)(1/2) \\
\mathrm{P}(Z_1 = -1 \wedge Z_0 = 1) &= (1/3)(1/2) \\
\mathrm{P}(Z_1 = 1 \wedge Z_0 = -1) &= 0 \\
\mathrm{P}(Z_1 = -1 \wedge Z_0 = -1) &= (1/3)(1/2)
\end{aligned}$$

thus autocorrelation of one time step must be positive:

$$\mathrm{E}[Z_1 Z_0] = (1 \cdot 1)\frac{1}{6} + (-1 \cdot 1)\frac{1}{6} + (-1 \cdot -1)\frac{1}{6} = \frac{1}{6}$$

For two time steps

$$\begin{aligned}
\mathrm{P}(Z_2 = 1 \wedge Z_0 = 1) &= (1/3)(1/2)^2 \\
\mathrm{P}(Z_2 = -1 \wedge Z_0 = 1) &= (1/3)[(1/2)^2 + (1/2)^2] \\
\mathrm{P}(Z_2 = 1 \wedge Z_0 = -1) &= (1/3)(1/2)^2 \\
\mathrm{P}(Z_2 = -1 \wedge Z_0 = -1) &= (1/3)(1/2)^2
\end{aligned}$$

thus the autocorrelation for two time steps must be negative:

$$\mathrm{E}[Z_2 Z_0] = (1 \cdot 1)\frac{1}{12} + (-1 \cdot 1)\frac{2}{12} + (1 \cdot -1)\frac{1}{12} + (-1 \cdot -1)\frac{1}{12} = -\frac{1}{12}$$

thus

$$\mathrm{Cor}[Z_2, Z_0] \neq \mathrm{Cor}[Z_2, Z_1]\,\mathrm{Cor}[Z_1, Z_0]$$

References

1. Hamilton JD (1994) Time series analysis. Princeton University Press, Princeton, N.J