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Short simple proof of the Cauchy-Schwarz inequality

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In my opinion, the following proof is the shortest, simplest and best proof of the Cauchy-Schwarz inequality. It is a proof developed in The Cauchy-Schwarz Master Class: An Introduction to the Art of Mathematical Inequalities [1]. Below are three variations of the proof at three increasing levels of abstraction. These three variations are expressed respectively in terms of:

- random variables [2]
- vectors of any *real* inner product space [1]
- vectors of *any* inner product space (real or complex) [3]

The Cauchy-Schwarz inequality was originally expressed in terms of sequences of numbers [1]. The continuous analogue is in terms of two integrable functions [4].

In terms of random variables

Given any two random variables X and Y,

$$\mathrm{E}[XY]^2 \leq \mathrm{E}[X^2]\mathrm{E}[Y^2]$$

with equality holding iff $aX + bY = 0$ for some constants a and b, at least one non-zero (i.e. X and Y are linearly dependent).

Proof

If either $\mathrm{E}[X^2] = 0$ or $\mathrm{E}[Y^2] = 0$ then $\mathrm{E}[XY] = 0$. Otherwise define

$$\hat{X} := \frac{X}{\sqrt{\mathrm{E}[X^2]}} \text{ and } \hat{Y} := \frac{Y}{\sqrt{\mathrm{E}[Y^2]}}$$

for which $\mathrm{E}\left[\hat{X}^2\right] = \mathrm{E}\left[\hat{Y}^2\right] = 1$. The proof follows from the product of two numbers always being less than or equal to the average of their squares

$$\begin{aligned} 0 &\le \mathrm{E}\left[(\hat{X} - \hat{Y})^2\right] \\ \mathrm{E}\left[\hat{X}\hat{Y}\right] &\le \mathrm{E}\left[\frac{\hat{X}^2 + \hat{Y}^2}{2}\right] \\ \frac{\mathrm{E}[XY]}{\sqrt{\mathrm{E}[X^2]}\sqrt{\mathrm{E}[Y^2]}} &\le \frac{1+1}{2} \\ \mathrm{E}[XY]^2 &\le \mathrm{E}\left[X^2\right]\mathrm{E}\left[Y^2\right] \end{aligned}$$

If both sides of the inequality are equal, linear dependence follows since either $X = 0$ or $Y = 0$ or

$$\frac{1}{\sqrt{\mathrm{E}[X^2]}}X + \frac{1}{\sqrt{\mathrm{E}[Y^2]}}Y = \hat{X} - \hat{Y} = 0$$

If X and Y are linearly dependent, either $X = kY$ or $Y = kX$ for some constant k, either way both sides of the inequality are equal.

QED

In terms of vectors of a *real* inner product space

The probabilistic proof can be generalized to any real inner product space as shown in [1].

Given any vectors x, y from a real inner product space, the Cauchy-Schwarz inequality is

$$\langle x, y \rangle \le \|x\| \, \|y\|$$

with equality holding iff x and y are linearly dependent.

Proof

If either $\|x\| = 0$ or $\|y\| = 0$ then $\langle x, y \rangle = 0$. Otherwise define

$$\hat{x} := \frac{x}{\|x\|} \quad \text{and} \quad \hat{y} := \frac{y}{\|y\|}$$

for which $\|\hat{x}\| = \|\hat{y}\| = 1$.

$$\begin{aligned} 0 &\le \langle \hat{x} - \hat{y}, \hat{x} - \hat{y} \rangle \\ 2\,\langle \hat{x}, \hat{y} \rangle &\le \langle \hat{x}, \hat{x} \rangle + \langle \hat{y}, \hat{y} \rangle \\ 2\,\frac{\langle x, y \rangle}{\|x\|\,\|y\|} &\le 1 + 1 \\ \langle x, y \rangle &\le \|x\|\,\|y\| \end{aligned}$$

If both sides of the inequality are equal, linear dependence follows since either $x = \vec{0}$ or $y = \vec{0}$ or

$$\frac{1}{\|x\|}x + \frac{1}{\|y\|}y = \hat{x} - \hat{y} = \vec{0}$$

If x and y are linearly dependent, either $x = \lambda y$ or $y = \lambda x$ for some scaler λ, either way both sides of the inequality are equal.

QED

In terms of vectors of an inner product space

This section considers the Cauchy-Schwarz inequality for vectors of a real or complex inner product space.

The proof for real inner produce spaces does not work for complex inner product spaces because $\langle y, x\rangle = \overline{\langle x, y\rangle}$ (complex conjugate).

The proof is effectively the same as the previous proof for real inner product spaces. But the normalized vectors $\hat{x}$ and $\hat{y}$ must be "rotated" in the complex plane so that both sides of the inequality remain real. This rotation will be done via a multiplier α.

Proof

Let $\hat{x}$ and $\hat{y}$ be defined as in the proof for real inner product spaces. If $\langle \hat{x}, \hat{y}\rangle = 0$ the inequality holds, otherwise let

$$\alpha := \sqrt{\frac{\langle \hat{y}, \hat{x}\rangle}{|\langle \hat{x}, \hat{y}\rangle|}}$$

for which the following convenient properties hold

$$\alpha\overline{\alpha} = 1 = \overline{\alpha}\alpha$$

$$\alpha^2 \langle \hat{x}, \hat{y}\rangle = \frac{\langle \hat{y}, \hat{x}\rangle \langle \hat{x}, \hat{y}\rangle}{|\langle \hat{x}, \hat{y}\rangle|} = \frac{|\langle \hat{x}, \hat{y}\rangle|^2}{|\langle \hat{x}, \hat{y}\rangle|} = |\langle \hat{x}, \hat{y}\rangle| = \overline{\alpha}^2 \langle \hat{y}, \hat{x}\rangle$$

The proof proceeds like with a real inner product space but using α,

$$\begin{aligned} 0 &\leq \langle \alpha\hat{x} - \overline{\alpha}\hat{y}, \alpha\hat{x} - \overline{\alpha}\hat{y}\rangle \\ &= \alpha\overline{\alpha} \langle \hat{x}, \hat{x}\rangle - \alpha^2 \langle \hat{x}, \hat{y}\rangle - \overline{\alpha}^2 \langle \hat{y}, \hat{x}\rangle + \overline{\alpha}\alpha \langle \hat{y}, \hat{y}\rangle \\ 2\,|\langle \hat{x}, \hat{y}\rangle| &\leq \langle \hat{x}, \hat{x}\rangle + \langle \hat{y}, \hat{y}\rangle \\ 2\,\frac{|\langle x, y\rangle|}{\|x\|\,\|y\|} &\leq 1 + 1 \\ |\langle x, y\rangle| &\leq \|x\|\,\|y\| \end{aligned}$$

If both sides of the inequality are equal, linear dependence follows since either $x = \vec{0}$ or $y = \vec{0}$ or

$$\frac{\alpha}{\|x\|}x + \frac{\overline{\alpha}}{\|y\|}y = \alpha\hat{x} - \overline{\alpha}\hat{y} = \vec{0}$$

If x and y are linearly dependent, either $x = \lambda y$ or $y = \lambda x$ for some scaler λ, either way both sides of the inequality are equal.

QED

References

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